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FMIAT: FREQUENCY MATRIX INCLUSION ANALYSIS TECHNIQUE FOR MCDM

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Abstract

The wider applicability of inclusion measures as well as the diversity of data existing in real life situations has motivated most of the researchers to introduce a two degree inclusion indicator within the intuitionistic fuzzy framework. Although these measures are capable of portraying the level of inclusion as well as a non inclusion relation existing between objects under consideration yet, they lose their practical application due to their computational complexity in many decision making situations. In our view, a single fuzzy degree intuitionistic inclusion measure will serve the purpose more effectively rather than a two degree inclusion measure for intuitionistic fuzzy sets. Therefore, in this research work we present a new yet effective technique to solve the Multi-Criteria Decision Making problems in intuitionstic fuzzy environment based on single degree inclusion measures called **The Frequency Matrix Inclusion Analysis Technique (FMIAT)**. This new technique is much simpler than any of the previously introduced techniques for MCDM in literature which makes it economically viable. The technique utilizes the *Parametric family of Fuzzy Inclusion Measures for IFS's* introduced in [1] as fundamental tool of analysis whose members in their own construction are based on variety of t-norms and t-conorms. This possible variation of t-norms and t-conorms gives our technique a clear advantage over other techniques as it becomes more flexible and can deal with all the three different states of mind (being Pessimist, Optimist and Neutral) of a decision maker by use of different operators in the respective

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inclusion indicators. Finally, an application of this technique is made in the field of Organizational Management.
Key words: intuitionistic fuzzy set (IFS), weighted average cardinality measure (WACM), fuzzy inclusion measure for intuitionistic fuzzy sets, MCDM (multi criteria decision making), FMIAT (frequency matrix inclusion analysis technique)

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1. Introduction and preliminaries. With fast globalization and inevitable economical competition among the enterprises, the need for an efficient performance appraisal has become the most important aspect of the human resource management. Any effective and comprehensive human resource appraisal is a multidimensional framework that deals with diverse factors and decisions that are accompanied by lots of uncertainties and vagueness involved in the quantification of many performance indices of an employee under judgment. Thus, for the past few decades, in order to achieve goals such as improvement in market competition and achievement of a flawless internal management that can attract and retain excellent employees, the researchers working in the field of management have dedicated themselves toward the search and development of the most dynamic, comprehensive yet computationally simple performance appraisal methodologies [2–5].

In recent years, the most desirable potential of an intuitionistic Fuzzy Set (IFS) to deal with the vagueness and uncertainties that exists in a real life situation, has drawn the attention of the researchers to construct different subsethood/inclusion measures in an Intuitionistic fuzzy environment. However, most of these measures have two degrees of inclusion indication which limit their practical applicability in a computational decision making scenario. Thus, to tackle this situation we believe that a single degree fuzzy inclusion indicator of IFS's that can fulfill the standard requirements for an inclusion measure, yet is simple in computations will serve the purpose of being the most reasonable, truly comparable and computationally economical inclusion indicator of intuitionistic fuzzy sets especially in the field of Multicriteria Decision Making in general and decision making in organizational management in particular.

Keeping all these factors in mind, in this work, we have introduced a new Multiciteria Decision making Technique called Frequency Matrix Inclusion Analysis Technique (FMIAT) that is based on single fuzzy degree inclusion measure of intuitionsitic sets. The family of fuzzy inclusion measures for intuitionistic fuzzy sets utilized in this work is called the *Parametric family of Fuzzy Inclusion Measures for IFS's* introduced in [1]. This new proposed technique is simple and economically viable. Moreover, it is flexible and can deal with all the three different states of mind (being Pessimist, Optimist and Neutral) of a decision maker by use of different operators in the respective inclusion indicators. A practical demonstration of this new technique is presented by applying it to the performance appraisal problem of the employees working in an organization.

Section 1 of this paper comprises of the basics of IFS theory utilized in this work while Section 2, and Subsection 2.1 is reserved for our new proposed technique for MCDM called FMIAT: Frequency Matrix Inclusion Analysis Technique and its case study, respectively.

Let $X = \{x_1, x_2, \dots, x_n\}$ be the universal set and $(\Upsilon, \leq_\Upsilon)$ be the complete bounded lattice given: $\Upsilon = \left\{\mu = (\mu_1, \mu_2) \in [0,1]^2 \mid \mu_1 + \mu_2 \leq 1\right\}$ with order $\leq_\Upsilon$ defined as $(\mu_1, \mu_2) \leq_\Upsilon (\nu_1, \nu_2)$ iff $\mu_1 \leq \nu_1$ and $\mu_2 \geq \nu_2$. This lattice Υ provides the mathematical foundation for upcoming definitions and results. The elements $1_\Upsilon = (1, 0)$ and $0_\Upsilon = (0, 1)$ are the greatest and the smallest element of the lattice Υ, respectively.

An intuitionistic fuzzy set *IFS* E on X is defined as a map $E : X \longrightarrow \Upsilon$ given by $E(x) = ((\delta_E(x), \eta_E(x)) = (\mu_1, \mu_2) \in \Upsilon$ satisfying $\mu_1 + \mu_2 \leq 1; \forall x \in X$. It however, may be defined alternatively by its founder as:

Definition 1.1 ([6]). Let X be the universe of discourse then an *intuitionistic fuzzy set* (*IFS*) is given by $E = \{(x, \delta_E(x), \eta_E(x)) \mid x \in X\}$ with functions $\delta_E(x)$ and $\eta_E(x) \in [0, 1]$ defining, respectively, the degree of membership and the degree of non membership of x in the set E satisfying the condition $(\forall x \in X)(\delta_E(x) + \eta_E(x) \leq 1)$. The collection of all IFS on X is denoted by $IFS(X)$.

Definition 1.2 ([1]). A *parametric family of fuzzy inclusion measures for intuitionistic fuzzy sets* is a class of maps $\Psi : IFS(X) \times IFS(X) \longrightarrow [0, 1]$ defined as:

$$\Psi(E, F) = \frac{\alpha(|E \backslash F|) + \beta(|F \backslash E|) + \gamma(|E \cap F|) + \lambda(|E \cup F)^c|)}{\beta(|E \triangle F|) + \gamma(|E \cap F|) + \lambda(|E \cup F)^c|)},$$

where α, β, γ, λ are positive real numbers. Because the values of $\Psi(E, F)$ should belong to $[0, 1]$, we have to impose an additional condition only on parameters α and β such that $0 \leq \alpha \leq \beta$.

Remark 1.3. In this work we have utilized the famous 5 members of the fuzzy inclusion measures extracted from the above parametric family of inclusion measures by setting different combinations of the parameters involved. Moreover we are using the notation Ψ^j_k, $k = 1, 2, 3, 4, 5$ to specify these transformed inclusion measures of the parametric family Ψ^j with subscript $j = M, P, L$ specifying the three Fuzzy Frank t-norms T_j given as $T_M(p, q) = \min(p, q)$; $T_P(p, q) = pq$ and $T_L(p, q) = \max(p + q - 1, 0)$; for all $p, q \in [0, 1]$; and S_j represents their corresponding dual t-conorms. In order to have a deep understanding of the expressions obtained below we invite our reader to consult [1]. However, for now we present the final expressions for the five members of the parametric family as shown in Table 1.1.

2. Frequency Matrix Inclusion Analysis Technique (FMIAT). In this section, we have developed a new and effective method for ranking of alternatives in a MCDM environment based on an inclusion technique. We have called this technique FMIAT: Frequency Matrix Inclusion Analysis Technique. Our proposed method of evaluation is based on the following steps:

Table 1.1

Measure	Expression	α	β	γ	λ
$\Psi_1^j(E,F)=\frac{\lvert F\setminus E\rvert}{\lvert E\triangle F\rvert}$	$\frac{\sum_{x\in X}\delta_F(x)-\eta_F(x)-T_j(\delta_E(x),\delta_F(x))+S_j(\eta_E(x),\eta_F(x))}{\sum_{x\in X}\delta_E(x)+\delta_F(x)-\eta_E(x)-\eta_F(x)-2T_j(\delta_E(x),\delta_F(x))+2S_j(\eta_E(x),\eta_F(x))}$	0	1	0	0
$\Psi_2^j(E,F)=\frac{\lvert E^c\rvert}{\lvert (E\cap F)^c\rvert}$	$\frac{2n-\sum_{x\in X}\delta_E(x)+1-\eta_E(x)}{2n-\sum_{x\in X}T_j(\delta_E(x),\delta_F(x))+1-S_j(\eta_E(x),\eta_F(x))}$	0	1	0	1
$\Psi_3^j(E,F)=\frac{\lvert F\rvert}{\lvert (E\cup F)\rvert}$	$\frac{\sum_{x\in X}\delta_F(x)+1-\eta_F(x)}{\sum_{x\in X}S_j(\delta_E(x),\delta_F(x))+1-T_j(\eta_E(x),\eta_F(x))}$	0	1	1	0
$\Psi_4^j(E,F)=\frac{\lvert (E\setminus F)^c\rvert}{n}$	$\frac{2n-\sum_{x\in X}\delta_E(x)-\eta_E(x)-T_j(\delta_E(x),\delta_F(x))+S_j(\eta_E(x),\eta_F(x))}{2n}$	0	1	1	1
$\Psi_5^j(E,F)=1-\frac{2\lvert E\setminus F\rvert}{n+\lvert (E\triangle F)\rvert}$	$\frac{2n+\sum_{x\in X}\delta_F(x)+(1-\eta_F(x))-\delta_E(x)-(1-\eta_E(x))}{2n+\sum_{x\in X}\delta_E(x)+\delta_F(x)-\eta_E(x)-\eta_F(x)-2T_j(\delta_E(x),\delta_F(x))+2S_j(\eta_E(x),\eta_F(x))}$	0	2	1	1

To begin with, we consider a particular set of schemes $E = \{E_1, E_2, E_3, \ldots, E_m\}$ that comprises m objects of evaluation/alternatives that are to be judged. Next, we shall specify another set called the Set of Indexes (criteria) of evaluation $G = \{G_1, G_2, G_3, \ldots, G_n\}$ for the formally selected set of alternatives E that will be judged on the basis of these criteria.

(i) Evaluation Matrix: On the basis of his expert opinion a decision maker will initially build an intuitionistic fuzzy relation between E and G in the form of an evaluation matrix presented below:

$$R = \begin{bmatrix} (G_1, \delta_{11}, \eta_{11}) & . & . & (G_p, \delta_{1p}, \eta_{1p}) & . & . & (G_n, \delta_{1n}, \eta_{1n}) \\ . & & & . & & & . \\ . & & & . & & & . \\ . & & & . & & & . \\ . & & & . & & & . \\ (G_1, \delta_{m1}, \eta_{m1}) & . & . & (G_p, \delta_{mp}, \eta_{msp}) & . & . & (G_n, \delta_{mn}, \eta_{mn}) \end{bmatrix}$$

Clearly, an intuitionistic fuzzy relation matrix R will elaborate the mathematical relationship between the scheme E_r with respect to the evaluation index G_t.

(ii) Super Set: From the set of schemes $E_r \in IFS(G)$, we construct an IFS set ψ called Super set given as:

$$\psi = \{(G_t, \delta_\psi(G_t), \eta_\psi(G_t)) \mid G_t \in G\} \tag{1}$$

such that $\delta_\psi(G_t) = \bigvee_{r=1}^{m} \delta_{Er}(G_t)$ and $\eta_\psi(G_t) = \bigwedge_{r=1}^{m} \eta_{E_r}(G_t)$.

Here, the fuzzy conjunction $\bigwedge$ and the fuzzy disjunction $\bigvee$ used in the formula can be modelled by any pair of fuzzy t-norms and their dual conorms.

(iii) Local Ranking: Utilizing the members of the family of inclusion measures Ψ_k^j from Table 1.1; for a fixed j and k ; $j = M, L, P$ and $k = 1, 2, \ldots, 5$ we define a local ranking order E_p dominates E_q as:

$$E_p \trianglerighteq E_q \text{ if } \Psi_k^j(E_p, \psi) \geq \Psi_k^j(E_q, \psi), \tag{2}$$

where $E_p, E_q \in IFS(G)$ and ψ is the super set defined in step (ii).

In this step, for a fixed j such that $j = M, P, L$ and using (2) we construct all possible chains of order for all $k = 1, 2, ..., 5$.

(iv) Global Ranking: We define a global ranking order of the alternatives based on the following steps:

(a) Frequency Matrix: Utilizing the data from the local ranking for a fixed j and for all $k = 1, 2, \ldots, 5$ we construct a crisp position frequency relation matrix Q_j between the set of alternatives E_r; $r = 1, 2, 3, \ldots, m$ and the set of positions P_i; $i = 1, 2, 3, \ldots, m$ such that each entry of the matrix q_{ri} is the count of the times the alternative E_r has attained the position P_i.

$$Q_j = \begin{bmatrix} q_{11} & \cdot & \cdot & & \cdot & \cdot & q_{1m} \\ \cdot & & & \cdot & & & \cdot \\ q_{r1}\cdot & & & q_{ri} & & & q_{rm} \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ q_{m1} & \cdot & \cdot & \cdot & \cdot & \cdot & q_{mm} \end{bmatrix}$$

(b) Final Ranking: In this second step, we define the final ranking of alternatives using the matrix Q_j from step (a). We scrutinize the highest value q_{r1} from the first column and we assign the corresponding alternative E_r the highest ranking position. Then, we exclude that row and column from the matrix meaning the position is now occupied by that alternative so he/she should now be excluded from the further process of evaluation. We repeat this process of selection from the matrix Q_j till all the alternatives are assigned their rightful positions.

(c) Ranking Tie: In case, there is a ranking tie between the alternatives in final ranking we propose two options:

(1) We look for the most repeated pattern in local ranking to handle the controversy between the positions.

(2) We can switch to a different selection of t-norm in step (iii).

2.1. Application of FMIAT to the management of an organization. The following subsection will elaborate the efficiency and simplicity of the new proposed Frequency Matrix Inclusion Analysis Technique by applying the proposed technique to an employee appraisal problem of an organizational management.

Let us work on the problem of the right decision for the selection of the most competent worker in an organization at the end of the year who can earn the benefits offered by its management. For the purpose let the selection criteria as set by the management be as follows:

G_1 : *Discipline;*
G_2 : *Punctuality;*
G_3 : *Team work;*
G_4 : *Progressiveness;*
G_5 : *Efficiency;*
G_6 : *Stress management.*

The short listed competitors/employees under consideration are:
E_1 : Kate;
E_2 : Mark;
E_3 : Philip;

E_4 : Theresa.

Now, to proceed with the selection process we follow the technique stepwise as:
(i) Evaluation Matrix: As discussed in the proposed technique the first is the construction of an Evaluation Matrix that will represent the four employees who are to be judged on the basis of all the six characteristics:

$$R = \begin{bmatrix} (G_1, 0.2, 0.7) & (G_2, 0.5, 0.2) & (G_3, 0.8, 0.1) & (G_4, 0.6, 0.3) & (G_5, 0.4, 0.5) & (G_6, 0.3, 0.6) \\ (G_1, 0.6, 0.2) & (G_2, 0.2, 0.7) & (G_3, 0.7, 0.3) & (G_4, 0.8, 0.2) & (G_5, 0.5, 0.3) & (G_6, 0.9, 0.1) \\ (G_1, 0.2, 0.7) & (G_2, 0.4, 0.5) & (G_3, 0.8, 0.2) & (G_4, 0.9, 0.1) & (G_5, 0.6, 0.3) & (G_6, 0.5, 0.2) \\ (G_1, 0.5, 0.4) & (G_2, 0.3, 0.5) & (G_3, 0.6, 0.3) & (G_4, 0.5, 0.3) & (G_5, 0.7, 0.2) & (G_6, 0.9, 0.0) \end{bmatrix}$$

(ii) Super Set: We now build the super intuitionistic fuzzy set ψ from the above listed data using equation (1). Among various options for a t-norm and its dual t-conorm in (1) in this example we will restrict ourselves to model the pair $\left(\bigwedge, \bigvee\right)$ by fuzzy (T_M, S_M) with their expressions given in Remark 1.3.
However, for this particular combination of t-norm and t-conorm in (1) our super set ψ becomes the intuitionistic fuzzy topological operator "closure" defined in [7].

$$\psi = \{(G_1, 0.6, 0.2), (G_2, 0.5, 0.2), (G_3, 0.8, 0.1), (G_4, 0.9, 0.1), (G_5, 0.7, 0.2), (G_6, 0.9, 0.0)\}.$$

(iii) Local Ranking: Next, we utilize the expressions of our inclusion measures given in Table 1.1??? and equation (2) to get the local ranking. Clearly, for our proposed technique, we have fixed the universe of discourse $X = G$: the set of index/criteria of evaluation, $E = E_r$; $r = 1, 2, 3, 4$ and $F = \psi$ in expressions given in Table 1.1.

For $j = P$ and for all $r = 1, 2, 3, 4$ we have the following local rankings:

Table 2.1.1

$\Psi_1^P(E_r, \psi)$	$E_1 \rhd E_3 \rhd E_4 \rhd E_2$
$\Psi_2^P(E_r, \psi)$	$E_1 \rhd E_3 \rhd E_4 \rhd E_2$
$\Psi_3^P(E_r, \psi)$	$E_1 \rhd E_3 \rhd E_4 \rhd E_2$
$\Psi_4^P(E_r, \psi)$	$E_1 \rhd E_3 \rhd E_4 \rhd E_2$
$\Psi_5^P(E_r, \psi)$	$E_1 \rhd E_3 \rhd E_2 \rhd E_4$

For $j = L$ and for all $r = 1, 2, 3, 4$ we have the following local rankings:

Table 2.1.2

$\Psi_1^L(E_r, \psi)$	$E_1 \rhd E_3 \rhd E_4 \rhd E_2$
$\Psi_2^L(E_r, \psi)$	$E_1 \rhd E_3 \rhd E_4 \rhd E_2$
$\Psi_3^L(E_r, \psi)$	$E_2 = E_3 \rhd E_1 \rhd E_4$
$\Psi_4^L(E_r, \psi)$	$E_1 = E_3 \rhd E_4 \rhd E_2$
$\Psi_5^L(E_r, \psi)$	$E_3 \rhd E_1 \rhd E_2 \rhd E_4$

In our given technique the selector is free to fix any one of the j's; $j = M, P, L$ to make the above calculations. In this case study, we have analyzed the given data by choosing $j = P, L$ only. The reason for not choosing $j = M$ is the fact that we have opted for model $\left(\bigwedge, \bigvee\right)$ in (1) by (T_M, S_M). This resulted into a situation where all of our alternatives E_r become complete subsets of super set ψ, hence cannot be ranked by choosing any of the inclusion measures Ψ_k^M; $k = 1, 2, 3, 4, 5$. However, a different choice of $\left(\bigwedge, \bigvee\right)$ in (1) would have allowed the calculation with $j = M$ also.

(iv) Global Ranking:

Frequency Matrix: We compute the position frequency relation matrices Q_j for $j = P, L$ as follows:

$$Q_P = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 1 \end{bmatrix}, \qquad Q_L = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 1 & 0 & 2 & 2 \\ 3 & 2 & 0 & 0 \\ 0 & 2 & 2 & 1 \end{bmatrix}$$

Final Ranking: Next, using step (b), we got the following global ranking of the alternatives from the matrix Q_p:
$E_1 \rhd E_3 \rhd E_4 \rhd E_2$ i.e., Kate $\rhd$ Philip $\rhd$ Theresa $\rhd$ Mark.
However, from matrix Q_L we have the following global ranking:
$E_1 = E_3 \rhd E_4 \rhd E_2$. Clearly, a tie of position 1 and 2 exists between the alternatives E_1 and E_3 while the other two alternatives are clearly positioned. In this situation, as mentioned in step (c) of global ranking we look into Table 2.1.2 for the most repeated pattern of positions for alternatives E_1 and E_3 and obtain the following ranking order from Ψ_1^L and Ψ_2^L:

$$E_1 \rhd E_3 \rhd E_4 \rhd E_2.$$

This is the same order as obtained by matrix Q_p which authenticates its validity.

Conclusion. In this work, we have employed the members of **Parametric family of Fuzzy Inclusion Measures for IFS's** introduced in [1] to design a new ranking technique for alternatives in MCDM environment called **The Frequency Matrix Inclusion Analysis Technique (FMIAT)**. This new technique is much simpler than any of the previously introduced techniques for MCDM in literature. We have executed this new technique to the field of organizational management for the selection of the best employee among the co-workers under a certain set of criteria. The new method gave remarkable results by providing the same ranking order of alternatives for different selections of t-norms and conorms. Moreover, the authenticity of the selected position for an alternative with respect to a choice of t-norm can be judged by taking a ratio between [the highest frequency corresponding to the selected alternative] and [the number of inclusion measures involved which in our case is 5] i.e., $\gamma_r = \dfrac{q_{ri}}{5}$. Clearly, $\gamma_r \in [0,1]$. Now if $\gamma_r = 1$ it means the authenticity of the achieved position for alternative E_r is 100%. For instance, in frequency matrix Q_P, the authenticity of the position 1 and 2 assigned to Kate and Philip is 100% while the authenticity of the position 3 and 4 assigned to Theresa and Mark is 80%. However, for a different set of data we shall obtain a different conclusion. But, as mentioned earlier whether we choose $j = P$ or $j = L$ the same ranking is obtained which allows the selector a free choice of measures depending upon his economical situation.

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